

# The Conformal Penrose Limit: Back to Square One

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## ABSTRACT

We show that the conformal Penrose limit is an ordinary plane wave limit in a higher dimensional framework which resolves the spacetime singularity. The higher dimensional framework is provided by Ricci-flat manifolds which are of the form  $M_D = M_d \times B$ , where  $M_d$  is an Einstein spacetime that has a negative cosmological constant and admits a spacelike conformal Killing vector, and  $B$  is a complete Sasaki-Einstein space with constant sectional curvature. We define the Kaluza-Klein metric of  $M_D$  through the conformal Killing potential of  $M_d$  and prove that  $M_d$  has a conformal Penrose limit if and only if  $M_D$  has an ordinary plane wave limit. Further properties of the limit are discussed.

# 1 Introduction

Recently, Penrose limit [1] and its gauge theory counterpart, the BMN limit [2], have played a pivotal role in understanding certain aspects of the AdS/CFT correspondence in string theory. This correspondence has its roots in  $AdS_{p+2} \times S^{D-p-2}$  type of geometries, which are products of an anti de Sitter ( $AdS$ ) spacetime and a sphere  $S$  of appropriate dimensions, and Penrose limits of  $AdS_{p+2} \times S^{D-p-2}$  spacetimes were found to be the maximally supersymmetric plane waves [3]. Moreover, in the central  $D = 10$ ,  $p = 3$  case, superstrings with non-trivial Ramond-Ramond fields could be consistently quantized on the plane wave background [4]. Further work based on these developments have furnished us with new insights about the AdS/CFT correspondence in a framework that surpasses the supergravity approximation.

Conformal Penrose limit [5],[6] is a new type of a limit, taken again in the vicinity of a null geodesic, which allows two properties to be preserved that were not permitted in the original Penrose limit. Conformal Penrose limit is designed to preserve a non-zero cosmological constant  $\Lambda$  and also takes into account the presence of metric functions homogeneous of degree zero in the coordinates. It turns out that this limit is available only when the spacetime admits a spacelike conformal Killing vector and  $\Lambda < 0$ . Whereas the Penrose limit always yields an ordinary plane wave, the conformal Penrose limits of such spacetimes turn out to be  $AdS$  plane waves. These  $AdS$  plane waves can be interpreted as the Randall-Sundrum zero mode [7], preserve 1/4 supersymmetries and possess a Virasoro symmetry [8]. The procedure works in all spacetime dimensions  $d \geq 4$  but in the case  $d = 4$ , it actually yields no wave degrees of freedom. This is not surprising because in  $d = 4$  the only spacetime that has  $\Lambda < 0$  and admits a spacelike conformal Killing vector (CKV) is the  $AdS$  space [9] and consequently, in this case the conformal Penrose limit amounts only to a symmetry of a unique space. This is perhaps the reason why the conformal limit was not taken into account in the original Penrose argument.

In string theory, conformal Penrose limit is relevant to the Freund-Rubin type of compactifications encountered in the study of AdS/CFT and DW/QFT dualities in various dimensions [10], [11]. When one considers the corresponding supergravity Lagrangians, one finds that the dilaton field must act as a potential for a CKV in the compactification process. This role of the dilaton was first utilized in the context of the  $D = 10$  dilatonic branes [12],[13] where the field equations were reduced to the Einstein equations with the appropriate cosmological constants. In general this type of reduction requires  $\Lambda < 0$  for the resulting lower,  $d$ -dimensional spacetime, but does not specify the type of CKV. Remark-

ably, the Freund-Rubin compactification does also require the CKV to be spacelike in the case of the  $D = 10$ ,  $p = 6$  Lagrangian, which is relevant to the  $D6$ -branes [6].

Although the  $AdS$  plane waves have various desirable properties, they are also known to suffer from pp-curvature singularities [14],[15]. At this singularity all scalar invariants of the  $AdS$  plane waves are well-behaved, but certain components of the Riemann tensor, relative to a frame which is parallelly transported along a causal geodesic, diverges. It is therefore of considerable interest to see whether the pp-curvature singularity can be resolved by some means in string theory. This issue was addressed in [6] and it was found that the singularity can be resolved only in the  $D = 10$ ,  $p = 6$  case by lifting up the limiting solution to  $D = 11$  supergravity. Recall that this was the only case where a condition on the type of the CKV was encountered.

Remarkably, what one gets in  $D = 11$  as the oxidation of the  $D = 10$ ,  $p = 6$  limiting solution is an ordinary plane wave, or an asymptotically locally Euclidean (ALE) plane wave with an  $A_{N-1}$  singularity [6]. This result raises in turn the question whether the conformal Penrose limit can be viewed always as an ordinary Penrose limit in a higher dimensional framework where the singularity is resolved. The purpose of the present paper is to furnish the framework in which this expectation is indeed fulfilled.

For this purpose we shall consider manifolds that are of the form  $M_D = M_d \times B$ , where  $M_d$  is a  $\Lambda \neq 0$  Einstein spacetime that admits a CKV and  $B$  is an internal space of appropriate dimension. The Kaluza-Klein (KK) metric of  $M_D$  will be taken to be conformal to the direct product of the metrics of  $M_d$  and  $B$ . We shall also require  $M_D$  to be Ricci-flat in a conformal gauge where the conformal factor of the metric is determined solely by the conformal Killing potential of  $M_d$ . The treatment will allow initially both of the signs for the pseudo-norm of the CKV as well as for  $\Lambda$ , and we shall see how spacelike CKV and  $\Lambda < 0$  conditions are simultaneously singled out together with the sign of the Ricci curvature of  $B$ . We shall note that almost all  $M_d$  of interest are singular at the fixed point of the CKV and find that these singularities are always resolved in the corresponding higher-dimensional  $M_D$  whenever  $B$  is a regular Sasaki-Einstein space. In order to avoid the presence of a scalar polynomial curvature singularity on  $M_D$ , which is not positioned at the fixed point of the CKV, the internal space  $B$  will be further restricted to the complete Sasaki-Einstein spaces of constant sectional curvature. We shall prove with this input that each  $M_d$  has a conformal Penrose limit if and only if the corresponding  $M_D$  has an ordinary plane wave limit.

Section 2 contains the proof for the case of hypersurface orthogonal CKV's. In this

case the higher dimensional spacetime turns out to be of a remarkably simple form:  $M_D = N \times C(B)$ , where  $N$  is the  $(d-1)$ -dimensional conformal boundary of  $M_d$  and  $C(B)$  is the flat cone over  $B$ . Due to this structure, a null geodesic of  $N$  that is passing from a fixed point of  $C(B)$  is a null geodesic of  $M_D$ . Taking the Penrose limit of  $M_D$  around such a geodesic with the help of the Penrose coordinates of  $N$  and the Kähler potential of  $C(B)$  gives a plane wave spacetime with at most a conical singularity. On  $M_d$  the same limit is then seen to be a conformal Penrose limit, giving an AdS plane wave. Section 3 presents the generalization of the argument to the CKV's which possess a non-zero twist and enables us to conclude that conformal Penrose limit can always be viewed as an ordinary Penrose limit in a higher dimension.

## 2 The Hypersurface Orthogonal Case

Let us keep the dimension  $D$  arbitrary and consider spacetimes that are of the form  $M_D = M_d \times B$ , where  $B$  is a  $(D-d)$ -dimensional Riemannian manifold. We shall use capital Latin letters  $M, N, \dots$  to label the tensor indices on  $M_D$ , the Greek letters  $\mu, \nu, \dots$  will refer to the coordinate bases of  $M_d$  and  $m, n, \dots$  will denote the coordinate indices on  $B$ . We shall assume that  $d \geq 4$  and the Lorentzian factor  $M_d$  is an Einstein space<sup>1</sup>:

$$R_{\mu\nu} = [\epsilon(d-1)/l^2]g_{\mu\nu}, \quad (2.1)$$

so that its cosmological constant is

$$\Lambda = [\epsilon(d-1)(d-2)/2l^2]. \quad (2.2)$$

Here  $l$  is a real parameter and  $\epsilon = \pm 1$  in order to allow  $\Lambda$  to take both signs. We shall also demand that  $M_d$  admits a smooth vector field  $V^\mu$  satisfying

$$\mathcal{L}_V g_{\mu\nu} = 2\psi g_{\mu\nu}, \quad (2.3)$$

where  $\mathcal{L}$  is the Lie derivative and  $\psi$  is a differentiable function on  $M_d$ . Then it can be deduced from (2.1) and (2.3) that  $\nabla_\mu \psi$  itself must be a hypersurface orthogonal CKV on  $M_d$ :

$$\nabla_\mu \nabla_\nu \psi = \frac{\epsilon}{l^2} \psi g_{\mu\nu}. \quad (2.4)$$

The properties of manifolds which admit an arbitrary CKV are well-known [16], and around any point with  $\nabla_\mu \psi \nabla^\mu \psi \neq 0$ , one can find a neighborhood where the metric  $g_{\mu\nu}$

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<sup>1</sup>Our spacetime conventions are same as [6]. We use in particular the mostly minus signature on  $M_D$ .

of  $M_d$  has a warped product form. In this neighborhood a coordinate system  $\{y, x^a\}$ ,  $a = 1, \dots, (d-1)$ , exists where  $\nabla_\mu \psi = U(y)\delta_\mu^y$ ,  $U = d\psi/dy$  and the line element takes the form:

$$ds_d^2 = \eta dy^2 + U^2(y)g_{ab}(x)dx^a dx^b. \quad (2.5)$$

Here  $g_{ab}(x)$  is a metric on a  $(d-1)$ -dimensional manifold  $N$  so that  $M_d = I \times_{U^2} N$ , where  $I$  is a real interval. Moreover,  $\eta = \pm 1$  is the sign of the pseudo-norm of the CKV:

$$\nabla_\mu \psi \nabla^\mu \psi = \eta U^2, \quad (2.6)$$

which is in general independent of the sign  $\epsilon$  of the cosmological constant. Notice that  $\eta = \pm 1$  also specifies whether  $N$  is a Riemannian or a Lorentzian manifold. When  $\epsilon = \eta = -1$ , the manifold  $N$  can be viewed as the conformal boundary of  $M_d$  [6].

The metrics which are of the form (2.5) and satisfy (2.1) constitute a two-parameter family of solutions which is described in detail in [6] and it is easy to see that  $M_d$  must be geodesically incomplete for almost all of these solutions. For example, the scalar invariant:

$$R_{\mu\nu\kappa\lambda} R^{\mu\nu\kappa\lambda} = 4(d-1)l^{-4} + U^{-4}\{2(d-1)(d-2)(U')^4 + 4\eta(U')^2 R_N + [R_{abcd} R^{abcd}]_N\}, \quad (2.7)$$

generically diverges at a zero of  $U$  which corresponds to a fixed point of the CKV. Here  $U' = dU/dy$ , the scalar curvature of  $N$  is denoted by  $R_N$  and in general, a subscript  $N$  on a quantity signifies that it is defined on  $N$ . The invariant (2.7) can be shown to be well-behaved at  $U = 0$  if  $\epsilon = \eta = -1$  and  $N$  is taken to be a Ricci-flat manifold whose all scalar invariants vanish. (A discussion of the structure of such  $N$ 's can be found in [17]). However, even in this case  $M_d$  will be incomplete unless  $N$  is flat. Since  $M_d$  has a conformal Penrose limit only when  $\epsilon = \eta = -1$ , it will be useful to study this subset in more detail.

Let us therefore specialize to  $\epsilon = \eta = -1$  and assume that  $N$  is a complete, Ricci-flat manifold. Suppose  $t^a = dx^a/d\tau$  is the unit tangent to a timelike geodesic of  $N$  with the affine parameter  $\tau$ . Let  $e_j^a$  be spacelike unit vectors,  $j = 1, \dots, d-2$ , such that  $(t^a, e_j^a)$  is an orthonormal basis of  $N$ , with the property that all  $e_j^a$  are parallelly transported along  $t^a$ . In order to construct a similar basis for  $M_d$ , let us next introduce the unit tangent  $t^\mu = dx^\mu/ds$  to a timelike geodesic of  $M_d$ . Then the two affine parameters will be related by  $ds/d\tau = c_0 U^2$ , where  $c_0$  is a non-zero real constant. Without any loss of generality one may choose  $c_0 = 1$  and for this choice it can be checked that the set of  $d$  unit vectors  $(t^\mu, e_j^\mu, e_{d-1}^\mu)$  defined by using the data on  $N$  as

$$t^\mu = \left\{ \frac{1}{U^2} t^a, \dot{y} \right\},$$

$$\begin{aligned} e_j^\mu &= \left\{ \frac{1}{U} e_j^a, 0 \right\}, \\ e_{d-1}^\mu &= \left\{ \frac{\dot{y}}{U} t^a, \frac{1}{U} \right\}, \end{aligned} \quad (2.8)$$

where  $\dot{y} = dy/ds$ , is the corresponding orthonormal basis for  $M_d$  which is parallelly transported along  $t^\mu$ . Here  $y$  is subject to  $\dot{y}^2 = U^{-2} - 1$ , because of the geodesic equation.

If one now examines the components of the Riemann tensor of  $M_d$  relative to the basis (2.8), one finds, for example, that

$$t^\mu e_j^\nu e_k^\kappa e_l^\lambda R_{\mu\nu\kappa\lambda} = \frac{1}{U^3} [t^a e_j^b e_k^c e_l^d R_{abcd}]_N, \quad (2.9)$$

and since  $[t^a e_j^b e_k^c e_l^d R_{abcd}]_N$  is perfectly well-behaved on  $N$ , it follows that (2.9) diverges at  $U = 0$  unless  $N$  is flat and this would imply that  $M_d = AdS_d$ . Hence for the present class of solutions  $M_d$  always suffers at least from a pp-curvature singularity which is located at a fixed point of the CKV unless  $M_d = AdS_d$ . As long as  $[R_{abcd} R^{abcd}]_N \neq 0$ , the singularity is in fact stronger because, (2.7) then exhibits a scalar polynomial singularity at  $U = 0$ .

Regardless of the nature of this singularity, each such  $M_d$  will have a conformal Penrose limit whose metric is [5]

$$d\hat{s}_d^2 = \frac{l^2}{z^2} [2dudv - h_{ij}(u)x^i x^j du^2 - \delta_{ij} dx^i dx^j - dz^2], \quad (2.10)$$

where  $z$  is a new coordinate ( $0 < z < \infty$ ) used in place of  $y$ , the range of the indices  $i, j$  is now  $i, j = 1, 2, \dots, d-3$ , and the metric functions satisfy:  $h_{jj}(u) = 0$ . (Here and in the sequel we use hats to distinguish the quantities that are the endpoints of the limits.) This shows that each such  $M_d$  has an AdS plane wave as a limit and although in general the presence of a singularity is not a hereditary property in the sense of [18], in our context it is preserved under the limit. Since (2.10) always has a pp-curvature singularity [14],[15] at the  $z = \infty$  fixed point of the CKV, what may not be inherited by the conformal Penrose limit is the type of the singularity.

Returning back to the  $D$ -dimensional picture, let us suppose that  $M_D$  is equipped with the metric:

$$ds_D^2 = (\ell/\psi)^2 [ds_d^2 + ds_B^2], \quad (2.11)$$

where  $ds_d^2$  and  $ds_B^2$  are the metrics of  $M_d$  and  $B$  respectively. Treating initially  $\epsilon$  and  $\eta$  as independent sign indicators, we also require  $M_D$  to be Ricci-flat :

$$R_{MN} = 0. \quad (2.12)$$

It then follows from (2.12), (2.1) and (2.4) that  $B$  must also be an Einstein space:

$$R_{mn} = [-\epsilon(D-d-1)/l^2] g_{mn}, \quad (2.13)$$

but with a cosmological constant that has an opposite sign, and that

$$g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi = \frac{\epsilon}{l^2}\psi^2. \quad (2.14)$$

Another consequence of our assumptions is that on  $M_D$  the components of the Riemann tensor  $R_{MNPQ}$  obey

$$R_{\mu m \nu n} = 0. \quad (2.15)$$

Conversely, if one starts from (2.12), (2.15), treats  $\psi$  in (2.11) as an arbitrary smooth scalar field on  $M_d$  and imposes (2.13), then the conditions (2.1), (2.4) and (2.14), which completely specify the type of  $M_d$  are obtained.

When  $M_D$  is constructed in this manner from the two-parameter family of  $M_d$ , the condition (2.14) together with (2.6) require that

$$\eta = \epsilon, \quad (2.16)$$

and consequently,  $\eta$  and  $\epsilon$  can no longer be independent. The same condition also requires

$$\psi^2 = l^2 U^2, \quad (2.17)$$

which is a constraint on the two parameters of the  $d$ -dimensional solutions, reducing the available  $M_d$  to the subset for which  $N$  is Ricci-flat. It follows that the metrics (2.5) that can be uplifted to  $M_D$  by the above procedure must be of the form

$$ds_d^2 = \frac{l^2}{z^2} [g_{ab}(x)dx^a dx^b + \epsilon dz^2], \quad (2.18)$$

with  $g_{ab}$  satisfying  $[R_{ab}]_N = 0$ . In terms of these coordinates,  $U = l/z$  and  $\psi = \pm l^2/z$ . When  $\epsilon = -1$  and  $N$  is taken to be the  $(d-1)$ -dimensional Minkowski space with  $x^a$  denoting the usual Minkowski coordinates, (2.18) reduces to the Poincaré patch of  $M_d = AdS_d$ .

Forming the  $D$ -dimensional metric (2.11) with this input then gives

$$ds_D^2 = g_{ab}(x)dx^a dx^b + \epsilon dz^2 + z^2 d\Omega^2, \quad (2.19)$$

where we have rescaled the metric on  $B$  as  $ds_B^2 = l^2 d\Omega^2$ , and relative to the (negative-definite) metric  $d\Omega^2$  the field equation for  $B$  is now:

$$R_{mn} = -\epsilon(D - d - 1)g_{mn}. \quad (2.20)$$

This shows that the  $D$ -dimensional Ricci-flat spacetimes that are constructed from Einstein manifolds admitting a CKV by the above procedure are necessarily of the form:

$$M_D = N \times C(B) \quad (2.21)$$

where  $C(B)$  is the Ricci-flat cone over  $B$ . When  $\epsilon = -1$ , the cone is Riemannian whereas  $\epsilon = 1$  implies that  $C(B)$  is Lorentzian, and in both cases  $z\partial/\partial z$  is an Euler vector field on  $C(B)$  generating an infinitesimal homothety. Such cones are known to play interesting roles in the context of AdS/CFT correspondence [19], [20].

One may view the above discussion as a KK reduction of the  $D$ -dimensional Ricci-flat theory to Einstein spaces  $M_d$  which is obviously a consistent reduction [21]. It would be desirable to maintain this consistency also in reductions to dimensions higher than  $d$  and a prerequisite for this behavior would be that  $B$  admits Killing vectors. Suppose  $B$  is compact and orientable. Since (2.20) must hold, it then follows from Bochner's argument [22] that, in order  $B$  to have isometries,

$$\epsilon = -1. \quad (2.22)$$

Due to this reason from now on we assume  $C(B)$  is a Riemannian cone.

Our next assumption about  $B$  is that it is a  $U(1)$  bundle over a  $(D - d - 1)$ -dimensional manifold  $K$ . This allows us to write

$$d\Omega^2 = d\bar{\Omega}^2 - (dY + \bar{A})^2, \quad (2.23)$$

where  $Y$  is the Killing coordinate,  $\bar{A}$  is the KK potential one-form and a bar over a quantity means that it is defined on  $K$ . When (2.22) and (2.23) are substituted into (2.20), the consistency of the  $(D - d - 1)$ -dimensional equations requires that  $K$  is Kähler and  $\bar{F} = d\bar{A}$  is related to the Kähler form  $\bar{w}$  of  $K$  by  $\bar{F} = 2\bar{w}$ . One then sees that the line element  $d\bar{\Omega}^2$  must obey

$$\bar{R}_{\alpha\beta} = (D - d + 1)\bar{g}_{\alpha\beta}, \quad (2.24)$$

where  $\alpha, \beta, \dots$  are the tensor indices on  $K$ . Hence  $K$  must be an even-dimensional, Kähler-Einstein manifold with positive Ricci curvature to maintain consistency. This result in turn implies that  $B$  must be a regular Sasaki-Einstein manifold and consequently, the metric cone  $C(B)$  is not only Ricci-flat but must also be Kähler, i.e. a Calabi-Yau cone. The properties of Sasaki-Einstein manifolds and their Kähler cones have been extensively studied [23], [24]. It is known in particular that  $\xi = J(z\partial/\partial z)$ , where  $J$  is the complex structure on  $C(B)$ , is the Reeb vector field which is both holomorphic and Killing. The one-form  $dY + \bar{A}$  is the contact form of  $B$  and is the dual to the vector field  $\xi$ . Moreover,  $z^2$  can be interpreted [24] as the Kähler potential of  $C(B)$ .

Assuming that  $N$  is complete, it is manifest in (2.19) that the  $z = \infty$  singularity of  $M_d$  is resolved on  $M_D$ . Unless  $B$  is taken to be an odd-dimensional unit sphere with the round metric, what one now has in the  $D$ -dimensional picture is a singularity at  $z = 0$  whose

nature depends crucially on the curvature of  $B$ . In the  $d$ -dimensional picture,  $z = 0$  is the locus of the conformal boundary  $N$  of  $M_d$  and is perfectly well-behaved. The corresponding  $M_D$ , however, suffers there from a scalar polynomial curvature singularity if the curvature of  $B$  is not constrained. One finds, for example, that the invariant:

$$R^{MOPQ}R_{MOPQ} = [R^{abcd}R_{abcd}]_N + (\ell/z)^4[R^{mnpq}R_{mnpq} - 2\ell^{-4}(D-d)(D-d-1)], \quad (2.25)$$

diverges at  $z = 0$  if the curvature of  $B$  does not render the second term to zero. One way to avoid the presence of scalar polynomial singularities on  $M_D$  is to demand that the internal space  $B$  has the minimum non-trivial dimension. In three dimensions the universal cover of  $B$  is isomorphic to the standard Sasaki-Einstein metric of  $S^3$  and  $C(B)$  is always a flat cone. This situation is precisely what was encountered in the framework of  $D = 11$  supergravity theory [6]. More generally, the same requirement can be met by specializing to  $B$  that are complete and have constant sectional curvature. Killing-Hopf theorem then implies that

$$B = S^{D-d}/\Gamma, \quad (2.26)$$

where  $\Gamma$  is a freely acting discrete subgroup of  $O(D-d+1)$ . With the choice (2.26) all curvature invariants of  $M_D$  reduce to those of  $N$  and since  $C(B)$  is again a flat cone, one has at most a conical singularity at  $z = 0$ . It is known that if  $B$  is complete, then  $C(B)$  is either flat or has irreducible holonomy [25] and the absence of scalar polynomial curvature singularities leaves out many interesting Sasaki-Einstein spaces when the dimension is not minimal. When  $B$  is the round unit sphere  $M_D$  has, of course, no singularity.

Consider now the ordinary Penrose limits of  $M_D$ . Since  $M_D = N \times C(B)$ , the set of all null geodesics of  $M_D$  can be viewed as the union of two disjoint subsets. In the first subset one has the null geodesics of  $N$  that are passing from fixed points of  $C(B)$  and the second subset is composed of the null geodesics which have one-dimensional traces on  $C(B)$ . The second subset can be viewed as the geodesics of  $C(B)$  plus the timelike geodesics of  $N$ . Suppose we choose a null geodesic from the first subset and apply the Penrose limit to its neighborhood. Then the Penrose coordinates of  $N$  together with the Kähler potential of  $C(B)$  are sufficient to specify the  $D$ -dimensional scaling rules. In addition to the standard Penrose scalings [1] on  $N$ , what one needs is to impose that the Kähler potential of  $C(B)$  scales according to

$$z \rightarrow \Omega_0 z, \quad (2.27)$$

where  $\Omega_0$  denotes the scaling parameter. Conformally rescaling the metric of (2.19) as  $\check{g}_{MN} = \Omega_0^{-2}g_{MN}$  and taking the limit  $\Omega_0 \rightarrow 0$  then gives

$$d\hat{s}_D^2 = 2dudv - h_{ij}(u)x^i x^j du^2 - \delta_{ij}dx^i dx^j - dz^2 + z^2 d\Omega^2, \quad (2.28)$$

which shows that the limiting spacetime is a particular  $D$ -dimensional plane wave spacetime for which the wave degrees of freedom of  $M_D$  coincide with that of  $N$  and a conical singularity at  $z = 0$  is allowed.

Notice that in (2.19) the dependence on the parameter  $\ell$  has completely disappeared. This is to be interpreted as a conformal gauge choice. Since the Ricci-flatness condition is preserved under the homotheties, it is clear that  $\ell$  can appear as a constant conformal factor in other conformal gauges. In the  $D$ -dimensional picture its scaling rule:

$$\ell \rightarrow \Omega_0 \ell, \quad (2.29)$$

can be inferred in the chosen conformal gauge by demanding that the conformal factor  $\ell/\psi$  of (2.11) remains invariant under the Penrose scalings .

Since  $M_d = I \times_{U^2} N$ , the null geodesic of  $N$  that was used to reach (2.28) can be viewed also as a null geodesic of  $M_d$  which is passing from a fixed point of  $I$ . Taking the conformal Penrose limit around such a geodesic of  $M_d$  involves precisely the same scalings that were employed on  $M_D$ . We therefore conclude that  $M_d$  has a conformal Penrose limit (2.10) if and only if  $M_D$  has the plane wave limit (2.28).

### 3 Inclusion of the Twist of the CKV

In this section we wish to consider a generalization of the above discussion which takes into account the presence of a non-zero twist of the CKV. For this purpose the coordinate system of (2.5) is not a suitable starting point. We therefore proceed as in [5] and utilize the fact that one can locally find another metric  $\tilde{g}_{\mu\nu}$  on  $M_d$  which is conformal to the original metric of (2.1):

$$g_{\mu\nu} = W^{-2} \tilde{g}_{\mu\nu}, \quad (3.1)$$

and for which  $V^\mu$  is an ordinary Killing vector:  $\mathcal{L}_V \tilde{g}_{\mu\nu} = 0$ . Here  $W$  is a differentiable scalar field and the map (3.1) will be available as long as  $V^\mu$  has no fixed points in the neighborhood. Choosing the Killing coordinate as  $V^\mu = \delta_z^\mu$  and using the standard KK decomposition one can express the line element for  $\tilde{g}_{\mu\nu}$  in the form

$$d\tilde{s}_d^2 = g_{ab}(x^c) dx^a dx^b + \eta \lambda^2 (dz + \zeta)^2, \quad (3.2)$$

where  $x^a$  are the remaining coordinates,  $\eta \lambda^2(x^c) = \tilde{g}_{\mu\nu} V^\mu V^\nu$  so that  $\eta$  is again the sign of the pseudo-norm of the CKV and  $\zeta = \zeta_a dx^a$  is a KK one-form. The CKV will have a non-zero twist if and only if  $(dz + \zeta) \wedge d\zeta \neq 0$ .

Let us assume that the conformal factor satisfies, with respect to the new metric, the conditions:

$$\tilde{\nabla}_\mu \tilde{\nabla}_\nu W = 0, \quad \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu W \tilde{\nabla}_\nu W = \frac{\epsilon}{l^2}, \quad (3.3)$$

which ensure that  $\tilde{g}_{\mu\nu}$  is a Ricci-flat metric on  $M_d$ . Equivalently, these conditions imply that  $\nabla_\mu W^{-1}$  is a closed CKV for the Einstein metric  $g_{\mu\nu}$ . It is therefore possible to identify

$$\psi = \ell W^{-1}, \quad (3.4)$$

and check whether  $\psi$  is related to the conformal Killing potential  $\psi_V = \nabla_\mu V^\mu/d$  that is associated with  $V^\mu$ .

The equations (3.3) have the simple solution

$$W = z/\ell + \chi(x^a, \ell), \quad (3.5)$$

provided  $k_a = \zeta_a - \ell \nabla_a \chi$  is a Killing vector for the  $(d-1)$ -dimensional metric  $g_{ab}$  and

$$\eta \lambda^{-2} + k^a k_a = \epsilon. \quad (3.6)$$

Here  $k^a k_a = g_{ab} k^a k^b$  and  $\chi(x^a, \ell)$  is an arbitrary differentiable function which may possess terms that are homogeneous of degree zero in  $\ell$  and  $x^a$ . For this solution  $\psi_V = -\psi/\ell^2$  and using  $\psi_V$  in (2.11) rather than (3.4) only amounts to working in another  $\ell$ -dependent conformal gauge.

From (3.6) it follows that, in order to allow a specialization to  $k_a = 0$  (or to  $k^a k_a = 0$ ) in the relevant solutions, one must require  $\eta = \epsilon$ . When (3.3) holds and the metric of  $M_D$  is constructed according to (2.11) and (3.4), the field equation (2.12) continues to imply (2.13). By the same assumptions on  $B$  one again ends up with (2.22) and the new form of the metric on  $M_d$  does not, of course, alter the conclusion that  $B$  is a Sasaki-Einstein space. After taking these considerations into account and redefining  $z + \ell \chi$  as a new  $z$  coordinate, the Einstein metric of  $M_d$  can be cast into the form

$$ds_d^2 = \frac{l^2}{z^2} [g_{ab}(x) dx^a dx^b - \lambda^2 (dz + k)^2], \quad (3.7)$$

where  $k = \zeta - \ell d\chi$  and the Ricci-flatness of  $\tilde{g}_{\mu\nu}$ , or equivalently (2.1), requires that

$$\begin{aligned} \nabla_a (\lambda^3 f^{ab}) &= 0, \\ \lambda^3 f^{ab} f_{ab} + 4\Delta\lambda &= 0, \\ R_{ab} &= 2^{-1} \lambda^2 f_a^c f_{bc} - \lambda^{-1} \nabla_a \nabla_b \lambda, \end{aligned} \quad (3.8)$$

where  $f_{ab} = \nabla_a k_b - \nabla_b k_a$  and all quantities including the D'Alembertian  $\Delta = \nabla^a \nabla_a$  again refer to  $g_{ab}$ . These equations generalize the  $[R_{ab}]_N = 0$  result of the previous section but it should be noted that  $g_{ab}$  is no longer the metric on the conformal boundary of  $M_d$ . The boundary  $N$  is still located at  $z = 0$  but is now equipped with the metric  $[g_{ab}]_N = g_{ab} - \lambda^2 k_a k_b$ . It turns out that, if the twist of the CKV is non-zero,  $[g_{ab}]_N$  also has a non-vanishing Ricci tensor prior to the limit. The corresponding Ricci-flat metric of  $M_D$  is

$$ds_D^2 = g_{ab}(x) dx^a dx^b - \lambda^2 (dz + k)^2 + z^2 d\Omega^2, \quad (3.9)$$

where the line element  $d\Omega^2$  of  $B$  is again governed by (2.20). Since  $k^a$  is also a Killing vector for  $[g_{ab}]_N$ , it is possible to express all the quantities appearing in (3.9) in terms of fields defined solely on  $N$  or  $C(B)$ . Letting  $\lambda_N = [g_{ab}]_N k^a k^b$  and  $k_N = [g_{ab}]_N k^a dx^b$ , one finds that

$$ds_D^2 = ds_N^2 + \lambda_N dz^2 - 2k_N dz + ds_{C(B)}^2, \quad (3.10)$$

and consequently,  $z^2$  is still the Kähler potential of  $C(B)$  but the direct product metric form is no longer available.

The presence of a non-zero twist does not modify the singularity structures of these manifolds. An examination of the invariant  $R^{\mu\nu\lambda\kappa} R_{\mu\nu\lambda\kappa}$  of  $M_d$  shows that as long as  $\tilde{R}^{\mu\nu\lambda\kappa} \tilde{R}_{\mu\nu\lambda\kappa}$  is regular and non-zero,  $M_d$  suffers from a scalar polynomial curvature singularity at  $z = \infty$ . If all curvature invariants constructed from  $\tilde{g}_{\mu\nu}$  turn out to be zero, this should become a pp-curvature singularity. Regardless of its nature, the singularity at  $z = \infty$  is always resolved in the higher-dimensional  $M_D$ . Provided the curvature invariants of  $\tilde{g}_{\mu\nu}$  are well-behaved, the higher dimensional  $M_D$  can be singular only at  $z = 0$  and in order to avoid scalar polynomial curvature singularities, one must again choose  $B = S^{D-d}/\Gamma$ . With this choice the curvature invariants of  $M_D$  reduce to the invariants constructed solely from  $\tilde{R}_{\mu\nu\lambda\kappa}$  with respect to the metric  $\tilde{g}_{\mu\nu}$ .

After taking the conformal Penrose limit of  $M_d$  by using the gauge conditions and the scaling rules of [5], the limit of (3.7) can be brought to the form

$$ds_d^2 = \frac{l^2}{z^2} [2dudv - h_{ij}(u) x^i x^j du^2 - \delta_{ij} dx^i dx^j - \lambda^2 (dz + \hat{k})^2], \quad (3.11)$$

where  $\lambda = \lambda(u)$  and  $\hat{k} = (\dot{b}_j x^j - l \dot{c}) du - b_j dx^j$  with  $b_j = b_j(u)$ ,  $c = c(u)$  and a dot denotes differentiation with respect to the null coordinate  $u$ . The field equation (2.1) and the conditions (3.3) are fully satisfied provided that

$$\lambda^{-2} + b_j b_j = 1, \quad (3.12)$$

$$\ddot{b}_j = h_{jk} b_k, \quad (3.13)$$

$$\ddot{c} = 0, \quad (3.14)$$

and

$$h_{jj} = -\ddot{\lambda}/\lambda - 2\lambda^2 \dot{b}_j \dot{b}_j. \quad (3.15)$$

These equations imply that the limit of  $g_{ab}$  is a plane wave metric which is not Ricci-flat whereas  $N$  is now equipped with a pp-wave metric that is Ricci-flat.

Since (3.11) is obtained by using the Penrose coordinates of  $\tilde{g}_{\mu\nu}$  and the associated null geodesics of  $M_d$  correspond to the null geodesics of  $M_D$  that are passing from fixed points of  $B$ , what has been accomplished in the  $D$ -dimensional picture is an ordinary Penrose limit giving

$$d\hat{s}_D^2 = 2dudv - h_{ij}(u)x^i x^j du^2 - \delta_{ij}dx^i dx^j - \lambda^2(dz + \hat{k})^2 + z^2 d\Omega^2. \quad (3.16)$$

Noting that this is again the metric of a plane wave spacetime with at most a conical singularity at  $z = 0$ , we conclude that conformal Penrose limit can be viewed as an ordinary plane wave limit in a higher dimension even when the CKV is not hypersurface orthogonal.

## 4 Discussion

It is well known that the plane waves owe their universal status as Penrose limits of general spacetimes to the existence of null geodesics. When one blows up a conjugate point-free neighborhood of such a geodesic of a given spacetime uniformly through the Penrose procedure, a plane wave spacetime results. In spacetime dimensions greater than four further care, however, must be exercised if the initial spacetime admits a CKV as well as a non-zero cosmological constant. It has been realized for some time that if  $\Lambda < 0$  and the CKV is spacelike, there is a distinguished class of null geodesics on such spacetimes which allows a more general, conformal Penrose limit. The neighborhoods of these geodesics can be blown in such a way that preserves the  $\Lambda \neq 0$  condition and one then ends up with AdS plane waves. In this paper we have seen that the conformal Penrose limit can be viewed as an ordinary Penrose limit in a higher dimension. Conversely, we have found that certain Penrose limits can be interpreted as conformal Penrose limits in lower dimensions, and it can be concluded that the dimensional reduction and oxidation processes commute with the limiting procedures in the present framework.

In this framework two crucial roles were played by the conformal Killing potential  $\psi$  and the conformal boundary  $N$  of  $M_d$ . Assuming that  $M_D = M_d \times B$ , we have considered

the whole conformal class of the metrics on  $M_D$  and demanded  $g_{MN}$  to be Ricci-flat in the conformal gauge of (2.11). In the context of string theory this gauge choice corresponds to working in the the dual frame [10]. Treating initially the signs of  $\Lambda$  and the pseudo-norm of the CKV as independent and arbitrary, the higher dimensional framework elucidated why the  $\Lambda < 0$ , spacelike CKV case must be singled out. Assuming that the CKV is hypersurface orthogonal, it is now clear that the Ricci-flatness of  $M_D$  in the chosen conformal gauge forces the signs of  $\Lambda$  and the pseudo-norm of the CKV to be the same,  $M_D$  to have the form  $M_D = N \times C(B)$  and  $\Lambda < 0$  ensures that  $C(B)$  is a Riemannian, Ricci-flat cone over a Sasaki-Einstein  $B$ . In this case  $N$  is Lorentzian and therefore possesses null geodesics which can be elevated to the null geodesics of either  $M_d$  or  $M_D$ . It is precisely these geodesics which allow one to map one limit into the other through oxidation or reduction. When  $\Lambda > 0$  these null geodesics are no longer available and consequently, conformal Penrose limit can never give rise to a dS plane wave rather than an AdS plane wave.

In the higher dimensional picture the conformal Killing potential, more precisely  $\ell^4\psi^{-2}$ , takes the role of the Kähler potential of  $C(B)$  and thereby allows one to infer the higher dimensional scaling rules from those of  $M_d$ . Taking the ordinary Penrose limit of  $M_D$  gives in general the plane wave limit  $\hat{N}$  of  $N$  times a Calabi-Yau cone:  $\hat{M}_D = \hat{N} \times C(B)$  and even in this general setting there is a remarkable dual singularity structure on  $M_d$  and  $M_D$ . The CKV fixed point singularity of  $M_d$  is always resolved on  $M_D$  but now  $M_D$  turns out to be singular at the apex of the Calabi-Yau cone. The cone singularity of  $M_D$  is in turn resolved in the lower dimensional picture since it just corresponds to the locus of the conformal boundary of  $M_d$ . Although it is an interesting limit on its own,  $\hat{M}_D$  with this general form is obviously not a plane wave spacetime. One requirement to end up with genuine plane waves in  $D$ -dimensions would be the vanishing of all the scalar polynomial curvature invariants in the limit and this was ensured by specializing to  $B = S^{D-d}/\Gamma$ . Prior to the limit, the same specialization had the virtue of eliminating all the scalar polynomial singularities of  $M_D$ .

In general a CKV can carry the degrees of freedom coded in its twist, in addition to its conformal Killing potential, and we have seen how the presence of a non-zero twist generalizes the cone structure of  $M_D$  and the metric on the conformal boundary of  $M_d$ . We have found that a CKV which has all the available degrees of freedom also allows the same mapping between the conformal and the ordinary Penrose limits. We feel that the implications of this mapping to the AdS/CFT and DW/QFT dualities, especially for the  $D=11$ ,  $d=8$  case, merit further investigation. For this case the underlying framework would

be furnished by  $D=11$  supergravity together with the  $SU(2)$  gauged,  $d = 8$  supergravity and ungauged  $d = 7$  supergravity theories [6].

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## References

- [1] R. Penrose, "Any space-time has a plane wave as a limit" in: *Differential Geometry and Relativity*, eds. M. Cahen and M. Flato (Reidel, Dordrecht, Netherlands 1976).
- [2] D. Berenstein, J.M. Maldacena and H. Nastase, JHEP **0204**, 013 (2002) [arXiv:hep-th/0202021].
- [3] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, Class. Quantum Grav. **19**, L87 (2002) [arXiv:hep-th/0201081].
- [4] R. R. Metsaev, Nucl. Phys. **B625**, 70 (2002) [arXiv: hep-th/0112044].
- [5] R. Güven, Phys. Lett. **B535**, 309 (2002) [arXiv: hep-th/0203153].
- [6] R. Güven, Class. Quantum. Grav. **23**, 295 (2006) [arXiv: hep-th/0508160].
- [7] A. Chamblin and G.W. Gibbons, Phys. Rev. Lett. **84**, 1090 (1999) [arXiv: hep-th/9909130].
- [8] M. Banados, A. Chamblin, G.W. Gibbons, Phys. Rev. **D61**, 08190 (2000) [arXiv: hep-th/9911101].
- [9] D. Garfinkel and Q. Tian, Class. Quantum. Grav. **4**, 137 (1987).
- [10] H.J. Boonstra, K. Skenderis and P.K. Townsend, JHEP **01**, 003 (1999) [arXiv: hep-th/9807137].
- [11] K. Behrndt, E. Bergshoeff, R. Halbersma and J.P. van der Schaar, Class. Quantum Grav. **16**, 3517 (1999) [arXiv: hep-th/9907006].
- [12] M.J. Duff, P.K. Townsend and P. van Nieuwenhuizen, Phys. Lett. **B122**, 232 (1983).

- [13] M.J. Duff, G.W. Gibbons and P.K. Townsend, Phys. Lett. **B332**, 321 (1994).
- [14] J. Podolský, Class. Quantum Grav. **15**, 719 (1998) [arXiv: gr-qc/9801052].
- [15] D. Brecher, A. Chamblin, H. Reall, Nucl. Phys. **B607**, 155 (2001) [arXiv: hep-th/0012076].
- [16] See e.g. W. Kühnel and H. Rademacher, Diff. Geom. and Appl. **7**, 237 (1997).
- [17] A. Coley, A. Fuster, S. Hervik and N. Pelavas, JHEP **0705**, 032 (2007) [arXiv: hep-th/0703256].
- [18] R. Geroch, Commun. Math. Phys. **13**, 180 (1969).
- [19] G.W. Gibbons, P. Rychenkova, Phys. Lett. **B443**, 138 (1998) [arXiv: hep-th/9809158].
- [20] B.S. Acharya, J.M. Figueroa-O'Farrill, C.M. Hull and B. Spence, Adv. Theor. Math. Phys. **2**, 1249 (1998) [arXiv: hep-th/9808014].
- [21] M.J. Duff and C.N. Pope, Nucl. Phys. **B255** 355 (1985); P. Hoxha, R.R. Martinez-Acosta and C.N. Pope, Class. Quantum Grav. **17**, 4207 (2000) [arXiv: hep-th/0005172].
- [22] S. Bochner, Bull. Amer. Math. Soc. **52**, 776 (1946); K. Yano, Ann. Math. **55**, 38 (1952).
- [23] See e.g. C.P. Boyer and K. Galicki, Supplemento ai Rendiconti del Circolo Matematico di Palermo Serie II. Suppl 75 (2005) 57, [arXiv: math.DG/0405256].
- [24] D. Martelli, J.S. Sparks, S.-T. Yau, "Sasaki-Einstein manifolds and volume minimisation", Preprint (2006) [arXiv: hep-th/0603021].
- [25] S. Gallot, Ann. Sci. École Norm. Sup. **12**, 235 (1979).